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BALANCING EXTENSIONS VIA BRUNN-MINKOWSKI

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We give a simple proof, based on the Brunn-Minkowski Theorem, of
Theorem. *In any finite poset P not a total order, there are elements x, y such that*

$$1/2e < p(x < y) < 1 - 1/2e.$$

A similar result was independently found by A. Karzanov and L. G. Khachiyan.

Introduction

For a finite partial order P and $x, y \in P$, denote by $p(x < y)$ the fraction of linear extensions of P in which x precedes y. (By a standard notational abuse we identify P with its element set. Linear extensions are defined below. For further background see e.g. [4].)

In this note we give a simple proof, based on the Brunn-Minkowski Theorem, of

Theorem 1. *In any finite poset P, not a total order, there are elements x, y such that*

$$1/2e < p(x < y) < 1 - 1/2e.$$

For a thorough discussion of this problem see [4], [6]. Let us mention that [4] gives a somewhat stronger bound, namely

$$3/11 < p(x < y) < 8/11,$$

but with a far more difficult proof. At this time no other proof is known of any bound of the form

$$\delta < p(x < y) < 1 - \delta$$

with δ a (positive) constant.

The conjectured best bound, $1/3 \le p(x < y) \le 2/3$ (M. Fredman circa 1975, [6]), remains open.

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After this paper was submitted we learned that A. Karzanov and L. G. Khachiyan [5] had somewhat earlier given a similar proof of the slightly weaker bound $1/e^2 < p(x < y) < 1 - 1/e^2$.

Review

A *linear extension* of P is an order preserving bijection

$$\pi : P \to \{1, \dots, n\},$$

where $n = |P|$. We write $E(P)$ for the set of such extensions, and $e(P)$ for its cardinality. For $x \in P$ the *average height* of x is

$$h(x) = \frac{1}{e(P)} \sum_{\pi \in E(P)} \pi(x).$$

It is easy to see that if P is not a total order then there are x, $y \in P$ with $|h(x) - h(y)| < 1$. So it's enough to show

Theorem 2. *If $|h(x) - h(y)| < 1$ then*

$$1/2e < p(x < y) < 1 - 1/2e.$$

Again, this is as in [4]. (As pointed out to us by T. Trotter, the bound of 3/11 in [4] is best possible for Theorem 2 in the sense that one can have (incomparable) x, y with $h(x) - h(y) = 1$ and $p(x < y) = 3/11$.)

Recall (our terminology follows [8]) that the *order polytope* $O(P)$ is the set of all $f \in \mathbf{R}^P$ satisfying

$$\begin{aligned} 0 \le f(x) \le 1 \qquad & \forall x \in P, \\ f(x) \le f(y) \quad \text{if} \quad & x \le y \quad \text{in} \quad P. \end{aligned}$$

Many of the combinational properties of P find natural expression in terms of $O(P)$.

With a linear extension π we associate the simplex

$$\textstyle\sum_\pi = \{f \in \mathbf{R}^P : 0 \le f(\pi^{-1}(1)) \le \dots \le f(\pi^{-1}(n)) \le 1\} \subseteq O(P).$$

We need the following easy facts (see e.g. [6]).

Lemma 3. (a) *The simplices $\sum_\pi$ triangulate $O(P)$.*
(b) $\mathrm{Vol}\,(O(P)) = |E(P)|/n!$
(c) *The centroid of $O(P)$ is $\frac{1}{n+1}h$.*

Our proof is based on the Brunn-Minkowski Theorem (e.g. [1]), and is inspired by the proof of the following result which was discovered more or less simultaneously by Grunbaum [3] and Hammer (unpublished), and rediscovered by Mityagin [7]. (We originally heard of Mityagin's paper from N. Megiddo, and subsequently of the earlier papers from L. Khachiyan. We understand from Professor Khachiyan that the result was probably known even before [3].)

Theorem. *Let K be a full-dimensional convex body in $\mathbf{R}^n$, $H = \{x : v \cdot x = 0\}$ a hyperplane through the centroid of K,*

and
$$H^+ = \{x : v \cdot x \geq 0\}.$$
Then
$$\mathrm{Vol}\,(K \cap H^+) \geq \left(\frac{n}{n+1}\right)^n \mathrm{Vol}\,K,$$
and in particular
$$\mathrm{Vol}\,(K \cap H^+) > \frac{1}{e}\mathrm{Vol}\,K.$$

Corollary. *If x, $y \in P$ satisfy $h(x) = h(y)$*

Then
$$\frac{1}{e} < p(x < y) < 1 - \frac{1}{e}.$$

This, surprisingly, is the same bound given by the (far more dificult) arguments of [4] in case $h(x) = h(y)$.

Proof of Theorem 2

Let x, $y \in P$ satisfy $|h(x) - h(y)| < 1$. It suffices to show $p(x > y) > 1/2e$. For $X \subseteq \mathbf{R}^P$, set
$$X_\lambda = \{f \in X : f(x) - f(y) = \lambda\},$$
$$X^+ = \{f \in X : f(x) - f(y) \geq 0\} = \bigcup_{\lambda \geq 0} X_\lambda,$$
and let c_X be the centroid of X. In particular, setting $K = O(P)$ we have

(1) $K_\lambda \neq \emptyset$ iff $\lambda \in [-1, 1]$,

(2) $c_K(x) - c_K(y) > -\frac{1}{n+1}$

(by Lemma 3(c)), and by Lemma 3(b) we are required to show

(3) $\mathrm{Vol}\,(K^+)/\mathrm{Vol}\,(K) > 1/2e$.

By the Brunn–Minkowski Theorem the function
$$r(\lambda) = \left[\frac{\mathrm{Vol}_{n-1}(K_\lambda)}{r_{n-1}}\right]^{\frac{1}{n-1}}$$
is concave on $[-1, 1]$, where r_{n-1} is the volume of the $(n-1)$-dimensional unit ball. Let B be the symmetrization of K with respect to the line $\ell = \mathbf{R}(e_x - e_y)$, where
$$\begin{aligned} e_x(z) &= 1 \quad \text{if} \quad z = x \\ &= 0 \quad \text{otherwise.} \end{aligned}$$
In other words,
$$B = \bigcup_{-1 \leq \lambda \leq 1} B_\lambda$$

where B_λ is the $(n-1)$-dimensional ball of radius $r(\lambda)$ centered at $\lambda(e_x - e_y)$ and contained in the hyperplane

$$\{f : f(x) - f(y) = \lambda\}.$$

Clearly $\mathrm{Vol}(B) = \mathrm{Vol}(K)$, $\mathrm{Vol}(B^+) = \mathrm{Vol}(K^+)$, so we just need (3) for B. Notice that B shares with K the properties

(4) $B_\lambda \neq \emptyset$ iff $\lambda \in [-1,1]$,

(5) $c_B(x) - c_B(y) = c_K(x) - c_K(y) > \frac{-1}{n+1}$.

At this point the situation is essentially 2-dimensional, and a picture will be helpful. Choose new coordinates $(x_1, \ldots, x_n)$ so that ℓ is the x_1-axis and B is obtained by rotating the curve $x_2 = r(x_1)$ about ℓ (see Fig. 1).

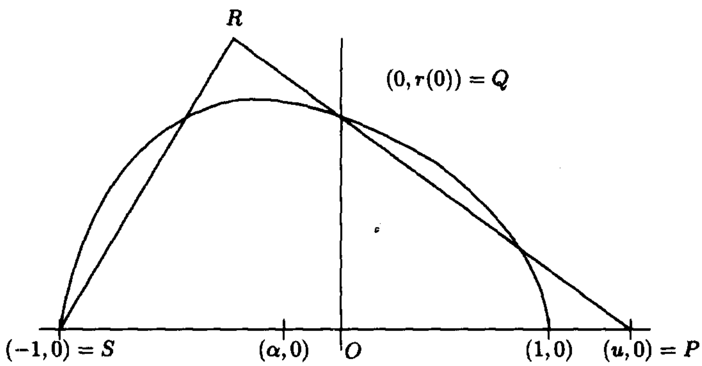

Fig. 1.

We will replace B by a double cone. Choose $u \geq 0$ so that, working in the (x_1, x_2)-plane and setting $P = (u,0)$, $Q = (0, r(0))$, $O = (0,0)$, the volume of the solid obtained by revolving the triangle PQO about ℓ is equal to $\mathrm{Vol}(B^+)$. Since $r(x_1)$ is concave, we have $u \geq 1$. Set $S = (-1,0)$ and choose R on the line $\overrightarrow{PQ}$ with Q between P and R so that the volume of the double cone D obtained by revolving the triangle PRS about ℓ is equal to $\mathrm{Vol}(B)$. Let $x_2 = s(x_1)$ be the function represented by the curve $\overrightarrow{PR} \cup \overrightarrow{RS}$.

Since for all t we have

$$\int_{-1}^{t} (r^{n-1}(x_1) - s^{n-1}(x_1))dx_1 \geq 0$$

(look at the picture!) and

$$\int_{-1}^{u} (r^{n-1}(x_1) - s^{n-1}(x_1))dx_1 = 0,$$

the x_1-coordinate of the centroid of D is at least as great as that of the centroid of B. It is thus enough to check

Proposition. *Let D be an n-dimensional double cone with apexes $(-1,0,\dots,0)$ and $(u,0,\cdots,0)$, and centroid $(\alpha,0,\dots,0)$. If $u \geq 1$ and $\alpha \geq -\frac{1}{n+1}$ then* $\mathrm{Vol}(D^+) > \frac{1}{2e}\mathrm{Vol}(D)$.

Proof. Let the two cones comprising D be C_1 and C_2, with apexes at $(-1,0,\dots,0)$ and $(u,0,\dots,0)$ respectively, and heights h_1 and h_2 (so $h_1+h_2=u+1$). Then

$$\frac{\mathrm{Vol}(D^+)}{\mathrm{Vol}(D)} = \frac{\mathrm{Vol}(D^+)}{\mathrm{Vol}(C_2)}\frac{\mathrm{Vol}(C_2)}{\mathrm{Vol}(D)} = \left(\frac{u}{h_2}\right)^n \frac{h_2}{u+1}. \tag{6}$$

This will be minimised when h_2 is as large as possible, so since h_2 evidently increases as α decreases, we may assume $\alpha = \frac{-1}{n+1}$.

The x_1-coordinates of the centroids of C_1 and C_2 are $-1+\frac{nh_1}{n+1}$ and $u-\frac{nh_2}{n+1}$ respectively. As

$$\frac{\mathrm{Vol}(C_1)}{\mathrm{Vol}(C_2)} = \frac{h_1}{h_2},$$

the centroid of D has x_1-coordinate

$$\frac{1}{h_1+h_2}\left[h_1\left(-1+\frac{nh_1}{n+1}\right)+h_2\left(u-\frac{nh_2}{n+1}\right)\right] = \frac{-1}{n+1}.$$

Solving for h_2 (using $h_1+h_2=u+1$) yields

$$h_2 = \frac{nu}{n-1},$$

which when substituted in (6) gives

$$\frac{\mathrm{Vol}(D^+)}{\mathrm{Vol}(D)} = \left(\frac{n-1}{n}\right)^{n-1}\frac{u}{u+1} > \frac{1}{2e}$$

(since $u \geq 1$). ∎

References

[1] H. Buseman: *Convex surfaces*, Interscience, New York, 1958.

[2] M. Fredman: How good is the information theory bound in sorting?, *Theoretical Computer Science* **1** (1976), 355–361.

[3] B. Grünbaum: Partitions of mass-distributions and of convex bodies by hyperplanes, *Pac. J. Math.* **10** (1960), 1257–1261.

[4] J. Kahn, and M. Saks: Balancing poset extensions, *Order* **1** (1984), 113–126.

[5] L. G. Khachiyan: Optimal algorithms in convex programming, decomposition and sorting, in: *Computers and Decision Problems* (Ju. Jaravlev, ed.), Moscow, Nauka, 1989, pp. 161–205 (Russian).

[6] N. Linial: The information theoretic bound is good for merging, *SIAM J. Comp.* **13** (1984), 795–801.

[7] B. S. MITYAGIN: Two inequalities for volumes of convex bodies, *Math. Notes* **5** (1968), 61–65.

[8] R. P. STANLEY: Two poset polytopes, *Discrete and Computational Geom.* **1** (1986), 9–23.

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